The Numerical Properties of G-heat equation and Related Application

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Abstract: In this paper, we consider the numerical convergence of G-heat equation which was first introduced by Peng. The G-heat equation extends the classical heat equation with uncertain volatility. For G-heat equation is nonlinear partial differential equation(PDE), we prove that the Newton iteration is convergence and the fully implicit discretization is monotone and stable. Then, we have the fully implicit discretization convergence to the viscosity solution of a G-heat equation.

Keywords: nonlinear PDE; G-heat equation; Newton iteration; fully implicit; viscosity solution.

1 Introduction

In recent years, a nonlinear expectation-G-expectation was established by Peng. In the theory of G-expectation, the G-normal distribution and G-Brownian motion were introduced and the corresponding stochastic calculus of Ito's type were established(see [5], [6], [7], [8], [9]). In Markovian case, the G-expectation is associated with fully nonlinear PDEs, which typical applications among economic and financial models with volatility uncertainty(see [3]).

Here, we study a kind very interesting nonlinear equation-G-heat equation, which is related to the G-normal distribution. G-normal distribution include distribution uncertain, and general the classical normal distribution. In economic and financial market, we usual assume some variables are classical normal distribution. But in many time it is fall, we need a new distribution to consider more complicated phenomenon. G-normal distribution is a new choice.

Under G-expectation, we relate the G-normal distribution to G-heat equation. Following the work of [3], [10], [11], [12], we use the technique in [3] to prove that the Newton iteration is convergence and the fully implicit discretization is monotone and stable.

In the last section. Under the same maximum volatility, we compare the value of G-normal distribution and classical normal distribution, and find that the values of G-normal distribution is larger more than classical normal distribution for the same event. So the G-normal distribution should be more prudent to measure risk. The further study see [2].

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The paper is organized as follows: in section 2, we give the notations and results on G-expectation. The numerical convergence to viscosity solution of fully implicit is established in section 3. Finally, we give an example to compare G-heat equation and heat equation in section 4.

2 Preliminaries

Firstly, we give the relation between G-heat equation and G-normal distribution.

Let Ω be a given set and let $\mathcal{H}$ be a vector lattice of real valued functions defined on Ω, namely $c \in \mathcal{H}$ for each constant c and $|X| \in \mathcal{H}$ if $X \in \mathcal{H}$. $\mathcal{H}$ is considered as the space of random variables.

Definition 2.1 *A sublinear expectation $\hat{\mathbb{E}}$ on $\mathcal{H}$ is a functional $\hat{\mathbb{E}} : \mathcal{H} \to R$ satisfying the following properties: for all $X, Y \in \mathcal{H}$, we have*

(a) Monotonicity: If $X \geq Y$ then $\hat{\mathbb{E}}[X] \geq \hat{\mathbb{E}}[Y]$;

(b) Constant preservation: $\hat{\mathbb{E}}[c] = c$;

(c) Sub-additivity: $\hat{\mathbb{E}}[X + Y] \leq \hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[Y]$;

(d) Positive homogeneity: $\hat{\mathbb{E}}[\lambda X] = \lambda \hat{\mathbb{E}}[X]$ for each $\lambda \geq 0$.

$(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a sublinear expectation space.

Definition 2.2 *Let X_1 and X_2 be two n-dimensional random vectors defined respectively in sublinear expectation spaces $(\Omega_1, \mathcal{H}_1, \hat{\mathbb{E}}_1)$ and $(\Omega_2, \mathcal{H}_2, \hat{\mathbb{E}}_2)$. They are called identically distributed, denoted by $X_1 \overset{d}{=} X_2$, if $\hat{\mathbb{E}}_1[\varphi(X_1)] = \hat{\mathbb{E}}_2[\varphi(X_2)]$, for all $\varphi \in C_{l.Lip}(R^n)$, where $C_{l.Lip}(R^n)$ is the space of real continuous functions defined on R^n such that*

$$|\varphi(x) - \varphi(y)| \leq C(1 + |x|^k + |y|^k)|x - y| \quad \text{for all } x, y \in R^n,$$

where k and C depend only on φ.

Definition 2.3 *In a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, a random vector $Y = (Y_1, \cdots, Y_n)$, $Y_i \in \mathcal{H}$, is said to be independent of another random vector $X = (X_1, \cdots, X_m)$, $X_i \in \mathcal{H}$ under $\hat{\mathbb{E}}[\cdot]$, denoted by $Y \perp X$, if for every test function $\varphi \in C_{l.Lip}(R^m \times R^n)$ we have $\hat{\mathbb{E}}[\varphi(X, Y)] = \hat{\mathbb{E}}[\hat{\mathbb{E}}[\varphi(x, Y)]_{x=X}]$.*

Definition 2.4 *(G-normal distribution) A d-dimensional random vector $X = (X_1, \cdots, X_d)$ in a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called G-normally distributed if for each $a, b \geq 0$ we have*

$$aX + b\bar{X} \overset{d}{=} \sqrt{a^2 + b^2} X,$$

where $\bar{X}$ is an independent copy of X, i.e., $\bar{X} \overset{d}{=} X$ and $\bar{X} \perp X$. Here the letter G denotes the function

$$G(A) := \frac{1}{2} \hat{\mathbb{E}}[\langle AX, X \rangle] : \mathbb{S}_d \to R,$$

where $\mathbb{S}_d$ denotes the collection of $d \times d$ symmetric matrices.

Peng [8] showed that $X = (X_1, \cdots, X_d)$ is G-normally distributed if and only if for each $\varphi \in C_{l.Lip}(R^d)$, $u(t,x) := \hat{\mathbb{E}}[\varphi(x + \sqrt{t}X)]$, $(t,x) \in [0,\infty) \times R^d$, is the solution of the following G-heat equation:

$$\partial_t u - G(D^2_{xx}u) = 0,\ u(0,x) = \varphi(x). \tag{2.1}$$

The function $G(\cdot) : \mathbb{S}_d \to R$ is a monotonic, sublinear mapping on $\mathbb{S}_d$ and $G(A) = \frac{1}{2}\hat{\mathbb{E}}[(AX, X)] \leq \frac{1}{2}|A|\hat{\mathbb{E}}[|X|^2] =: \frac{1}{2}|A|\bar{\sigma}^2$ implies that there exists a bounded, convex and closed subset $\Gamma \subset \mathbb{S}_d^+$ such that

$$G(A) = \frac{1}{2}\sup_{\gamma\in\Gamma} \mathrm{tr}[\gamma A],$$

where $\mathbb{S}_d^+$ denotes the collection of nonnegative elements in $\mathbb{S}_d$.

Let $d = 1$, we consider the finite difference method to the next G-heat equation:

$$\begin{aligned} &\partial_t u - \tfrac{1}{2}(\bar{\sigma}^2(D^2_{xx}u)^+ - \underline{\sigma}^2(D^2_{xx}u)^-) = 0, \quad x \in R, \quad t > 0, \\ &u(0,x) = \varphi(x), \quad x \in R, \end{aligned} \tag{2.2}$$

which

$$(D^2_{xx}u)^+ = \begin{cases} D^2_{xx}u, & D^2_{xx}u \geq 0, \\ 0, & D^2_{xx}u < 0, \end{cases}$$

and

$$(D^2_{xx}u)^- = \begin{cases} -D^2_{xx}u, & D^2_{xx}u \leq 0, \\ 0, & D^2_{xx}u > 0. \end{cases}$$

3 Numerical Convergence

Firstly, we consider a bounded boundary problem of (2.2), i.e.,

$$\begin{aligned} &\partial_t u - \tfrac{1}{2}(\bar{\sigma}^2(D^2_{xx}u)^+ - \underline{\sigma}^2(D^2_{xx}u)^-) = 0, \\ &u(0,x) = \varphi(x), \quad x \in [a,b], \\ &u(t,a) = g(t),\ u(t,b) = h(t) \quad t \in [0,T]. \end{aligned} \tag{3.1}$$

where φ, g, h are continuous functions.

3.1 A Finite Difference Discretization

The equation (3.1) can be discretized by a standard finite difference method with variable timeweighting to give

$$u_i^{n+1} - u_i^n = \quad \theta\alpha_i^n[u_{i+1}^n - 2u_i^n + u_{i-1}^n] + (1-\theta)\alpha_i^{n+1}[u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}], \tag{3.2}$$

where

$$
\begin{aligned}
\alpha_i^n &:= \tfrac{\sigma(\Gamma_i^n)^2 \triangle t_i}{2(x_{i+1}-x_i)(x_i-x_{i-1})}, \\
\sigma(\Gamma_i^n) &:= \begin{cases} \bar{\sigma}, & \text{if } \Gamma_i^n \geq 0 \\ \underline{\sigma}, & \text{if } \Gamma_i^n < 0 \end{cases}, \\
\Gamma_i^n &:= \tfrac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(x_{i+1}-x_i)(x_i-x_{i-1})}.
\end{aligned}
\tag{3.3}
$$

In this paper, we consider the fully implicit schemes with $\theta = 0$, i.e.,

$$
u_i^{n+1} - u_i^n = \alpha_i^{n+1}[u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}], \tag{3.4}
$$

The set of algebraic equation (3.1) is nonlinear for the formula of $\sigma(\Gamma_i^n)$. So we consider the discrete equation at each node as

$$
\varphi_i^n := u_i^n - u_i^{n+1} + \alpha_{i+1}^{n+1}[u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}].
$$

Following the work of D.M. Pooey [3] (more see, Pang and Qi [10];Qi and Sun [11]; Sun and Han [12]), we must specify the element of the generalized Jacobian that will be used in the Newton iteration. We define the derivatives as

$$
\frac{\partial \sigma(\Gamma)^2 \Gamma}{\partial \Gamma} = \begin{cases} \bar{\sigma}^2, & \text{if } \Gamma \geq 0 \\ \underline{\sigma}^2, & \text{if } \Gamma < 0 \end{cases},
$$

For further analysis the Newton iteration, we rewrite the discrete equation (3.2) in matrix form. Let

$$
\begin{aligned}
& U^{n+1} = [u_0^{n+1}, u_1^{n+1}, \cdots, u_m^{n+1}]', \quad U^n = [u_0^n, u_1^n, \cdots, u_m^n]', \\
& [M^n U^n]_i := -\alpha_i^n [u_{i+1}^n - 2u_i^n + u_{i-1}^n].
\end{aligned}
$$

For convenience, we modify the first and last rows of M as needed to handle the bounded boundary conditions. By the discretization schemes in (3.4), the matrix M is a diagonally dominant matric with positive diagonals and non-positive off-diagonals. Note that all the elements of the inverse of M are non-negative. The discrete equation (3.4) can be rewritten as:

$$
[I + M^{n+1}]U^{n+1} = U^n, \tag{3.5}
$$

where I is the identity matrix. Next we prove the convergence of the Newton iteration for full implicit schemes.

3.2 Convergence of the Newtion Iteration Schemes

For the matrix M is a diagonally dominant matric, we can analysis the Newton iteration of equation (3.5). We adopt the Newton timestep as the following scheme:

(a) Let $(U^{n+1})^0 = U^n$;

(b) For $k = 0, 1, 2, \cdots$ Solve

$$[I + M((U^{n+1})^k)](U^{n+1})^{k+1} = U^n \tag{3.6}$$

where $(U^{n+1})^{k+1}$ is the $(k+1)$th iteration, and $M((U^{n+1})^k)$ means M be dependent on $(U^{n+1})^k$.

(c) For a given small number ε, if

$$\max_i \left|(u_i^{n+1})^{k+1} - (u_i^{n+1})^k\right| < \varepsilon \cdot \max_i(1, \left|(u_i^{n+1})^{k+1}\right|).$$

(d) we end the scheme.

We give the convergence results about the above Newton iteration as followed:

Theorem 3.1 *The nonlinear iteration (3.6) convergence to the unique solution of (3.5), for given intial iterate* $(U^{n+1})^0 = U^n$.

Proof. For notional convergence, we denote $\hat{M}^k = M((U^{n+1})^k)$ and $\hat{U}^k = (U^{n+1})^k$. So equation (3.6) can be rewritten as

$$[I + \hat{M}^k]\hat{U}^{k+1} = U^n. \tag{3.7}$$

Firstly, we show that the sequence $\{\hat{U}^k\}_{0\le k}$ is monotonically. The k iteration of equation (3.7) gives

$$[I + \hat{M}^{k-1}]\hat{U}^k = U^n. \tag{3.8}$$

Subtracting equation (3.7) from equation (3.8), we have

$$[I + \hat{M}^k](\hat{U}^{k+1} - \hat{U}^k) = [\hat{M}^{k-1} - \hat{M}^k]\hat{U}^k. \tag{3.9}$$

We consider the right side of 3.9, for each i

$$([\hat{M}^{k-1} - \hat{M}^k]\hat{U}^k)_i = \frac{\triangle t_i(\sigma(\hat{\Gamma}_i^k)^2 - \sigma(\hat{\Gamma}_i^{k-1})^2)}{2}\hat{\Gamma}_i^k,$$

where

$$\begin{aligned}
&\hat{\Gamma}_i^k = \tfrac{\hat{u}_{i+1}^k - 2\hat{u}_i^k + \hat{u}_{i-1}^k}{(x_{i+1}-x_i)(x_i - x_{i-1})},\\
&\hat{U}^k := [\hat{u}_0^k, \hat{u}_1^k, \cdots, \hat{u}_m^k]',\\
&\sigma(\hat{\Gamma}_i^k) := \begin{cases} \bar{\sigma}, & \text{if } \hat{\Gamma}_i^k \ge 0 \\ \underline{\sigma}, & \text{if } \hat{\Gamma}_i^k < 0 \end{cases}.
\end{aligned}$$

By the equation (3.3), if $\hat{\Gamma}_i^k \le 0$, $\sigma(\hat{\Gamma}_i^k)^2 = \underline{\sigma}$, then

$$\frac{\triangle t_i(\sigma(\hat{\Gamma}_i^k)^2 - \sigma(\hat{\Gamma}_i^{k-1})^2)}{2}\hat{\Gamma}_i^k \ge 0;$$

Similarly, if $\hat{\Gamma}_i^k \ge 0$, $\sigma(\hat{\Gamma}_i^k)^2 = \bar{\sigma}$, then

$$\frac{\triangle t_i(\sigma(\hat{\Gamma}_i^k)^2 - \sigma(\hat{\Gamma}_i^{k-1})^2)}{2}\hat{\Gamma}_i^k \ge 0.$$

For the matric $I+\hat{M}^k$ is a diagonally dominant matric, the inverse of matric $I+\hat{M}^k$ is non-negative, we have

$$\hat{U}^{k+1}-\hat{U}^k \geq 0, \quad k \geq 1. \tag{3.10}$$

Next, we need to prove the sequence $\{\hat{U}^k\}_{0\leq k}$ is bounded. Set $C_{\max}=\max_i u_i^n$, $C_{\min}=\min_i u_i^n$, $\hat{U}_{\max}=\max_i \hat{u}_i^k$, $\hat{U}_{\min}=\min_i \hat{u}_i^k$. By the equation (3.8), we have

$$\hat{u}_i^k-\hat{\alpha}_i^{k-1}[\hat{u}_{i+1}^k-2\hat{u}_i^k+\hat{u}_{i-1}^k]=u_i^n, \tag{3.11}$$

where

$$\alpha_i^k:=\frac{\sigma(\hat{\Gamma}_i^k)^2\triangle t_i}{2(x_{i+1}-x_i)(x_i-x_{i-1})}. \tag{3.12}$$

By the equation (3.11), and $\hat{\alpha}_i^{k-1}\geq 0$, then

$$(1+2\hat{\alpha}_i^{k-1})\hat{u}_i^k \leq 2\hat{\alpha}_i^{k-1}\hat{U}_{\max}+C_{\max},$$

and

$$(1+2\hat{\alpha}_i^{k-1})\hat{u}_i^k \geq 2\hat{\alpha}_i^{k-1}\hat{U}_{\min}+C_{\min}.$$

So

$$\hat{u}_i^k \leq \frac{2\hat{\alpha}_i^{k-1}}{1+2\hat{\alpha}_i^{k-1}}\hat{U}_{\max}+C_{\max},$$

and

$$\hat{u}_i^k \geq \frac{2\hat{\alpha}_i^{k-1}}{1+2\hat{\alpha}_i^{k-1}}\hat{U}_{\min}+C_{\min}.$$

Set $\max_i \frac{2\hat{\alpha}_i^{k-1}}{1+2\hat{\alpha}_i^{k-1}}=b_1$, $\max_i \frac{2\hat{\alpha}_i^{k-1}}{1+2\hat{\alpha}_i^{k-1}}=b_2$, and $0\leq b_1, b_2<1$, then we have

$$\hat{U}_{\max}\leq \frac{1}{1-b_1}C_{\max}, \quad \hat{U}_{\min}\geq \frac{1}{1-b_2}C_{\min}.$$

Now, we prove the uniqueness. Suppose there are two solution to equation (3.5), U_1^{n+1} and U_2^{n+1}, such that

$$[I+M_1]U_1^{n+1}=U^n, \quad [I+M_2]U_2^{n+1}=U^n.$$

Similar the proof of the monotonicity sequence $\{\hat{U}^k\}_{0\leq k}$, we have

$$[I+M_2](U_2^{n+1}-U_1^{n+1})=[M_1-M_2]U_1^{n+1},$$

and

$$U_2^{n+1}-U_1^{n+1}\geq 0$$

By the equality of U_1^{n+1} and U_2^{n+1}, we have $U_2^{n+1}-U_1^{n+1}=0$.

Thus, we complete the proof.

3.3 The Convergence of Fully Implicit

In the above section. we have proved the convergence of the Newton iteration for the nonlinear equation (3.4). Next, we would to prove the full implicit schemes convergence to the viscosity solution of (3.1). By the work of Barles in [1], we know that a stable, consistent, and monotone discretization will convergence to the viscosity solution.

Theorem 3.2 *The fully implicit discretization (3.5) convergences to the solution of the equation (3.1), as* $\triangle t$, $\triangle x \to 0$.

We first give some important lemmas for prove Theorem 3.2.

Review the discrete equation at each node as

$$\varphi_i^n := u_i^n - u_i^{n+1} + \alpha_{i+1}^{n+1}[u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}] \tag{3.13}$$

then at each step

$$\varphi_i^n(u_{i+1}^{n+1}, u_i^{n+1}, u_{i-1}^{n+1}, u_i^n) = 0, \quad \forall i. \tag{3.14}$$

In the case fo nondifferentiable φ_i^n, we use the following definition of monotonicity:

Definition 3.3 *A discretization of the form (3.14) is monotone if either*

$$\begin{aligned}
&\varphi_i^n(u_{i+1}^{n+1} + \varepsilon_{i+1}^{n+1}, u_i^{n+1}, u_{i-1}^{n+1} + \varepsilon_{i-1}^{n+1}, u_i^n + \varepsilon_i^n) \geq \varphi_i^n(u_{i+1}^{n+1}, u_i^{n+1}, u_{i-1}^{n+1}, u_i^n),\\
&\varphi_i^n(u_{i+1}^{n+1}, u_i^{n+1} + \varepsilon_i^{n+1}, u_{i-1}^{n+1}, u_i^n) \leq \varphi_i^n(u_{i+1}^{n+1}, u_i^{n+1}, u_{i-1}^{n+1}, u_i^n),\\
&\forall \varepsilon_{i+1}^{n+1}, \varepsilon_i^{n+1}, \varepsilon_{i-1}^{n+1}, \varepsilon_i^n \geq 0,
\end{aligned}$$

or

$$\begin{aligned}
&\varphi_i^n(u_{i+1}^{n+1} + \varepsilon_{i+1}^{n+1}, u_i^{n+1}, u_{i-1}^{n+1} + \varepsilon_{i-1}^{n+1}, u_i^n + \varepsilon_i^n) \leq \varphi_i^n(u_{i+1}^{n+1}, u_i^{n+1}, u_{i-1}^{n+1}, u_i^n),\\
&\varphi_i^n(u_{i+1}^{n+1}, u_i^{n+1} + \varepsilon_i^{n+1}, u_{i-1}^{n+1}, u_i^n) \geq \varphi_i^n(u_{i+1}^{n+1}, u_i^{n+1}, u_{i-1}^{n+1}, u_i^n),\\
&\forall \varepsilon_{i+1}^{n+1}, \varepsilon_i^{n+1}, \varepsilon_{i-1}^{n+1}, \varepsilon_i^n \geq 0.
\end{aligned}$$

Next, we prove the monotonicity of the fully implicit discretization.

Lemma 3.4 *The fully implicit discretization (3.13) is monotone, indepdent of any choice of* $\triangle t$ *and* $\triangle x$.

Proof: For any given $\varepsilon > 0$, we just to chek the next two euation:

$$\begin{aligned}
&\varphi_i^n(u_{i+1}^{n+1} + \varepsilon, u_i^{n+1}, u_{i-1}^{n+1}, u_i^n) \geq \varphi_i^n(u_{i+1}^{n+1}, u_i^{n+1}, u_{i-1}^{n+1}, u_i^n),\\
&\varphi_i^n(u_{i+1}^{n+1}, u_i^{n+1} + \varepsilon, u_{i-1}^{n+1}, u_i^n) \leq \varphi_i^n(u_{i+1}^{n+1}, u_i^{n+1}, u_{i-1}^{n+1}, u_i^n).
\end{aligned}$$

By the definition of φ_i^n, we have

$$\begin{aligned}\varphi_i^n(u_{i+1}^{n+1}+\varepsilon, u_i^{n+1}, u_{i-1}^{n+1}, u_i^n) = & \quad u_i^n - u_i^{n+1} + \alpha_{i+1}^{n+1}[u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}] + \alpha_{i+1}^{n+1} \cdot \varepsilon \\ \geq & \quad u_i^n - u_i^{n+1} + \alpha_{i+1}^{n+1}[u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}] \\ = & \quad \varphi_i^n(u_{i+1}^{n+1}, u_i^{n+1}, u_{i-1}^{n+1}, u_i^n)\end{aligned}$$

and

$$\begin{aligned}\varphi_i^n(u_{i+1}^{n+1}, u_i^{n+1}+\varepsilon, u_{i-1}^{n+1}, u_i^n) = & \quad u_i^n - u_i^{n+1} + \alpha_{i+1}^{n+1}[u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}] - (2\alpha_{i+1}^{n+1}+1) \cdot \varepsilon \\ \leq & \quad u_i^n - u_i^{n+1} + \alpha_{i+1}^{n+1}[u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}] \\ = & \quad \varphi_i^n(u_{i+1}^{n+1}, u_i^{n+1}, u_{i-1}^{n+1}, u_i^n).\end{aligned}$$

This completes the proof.

Proof of Theorem 3.2:

By the results of Barles, we just to check that the fully implicit discretization is consistent, stable, monotone. Fristly, the formula (3.14) is a consistent discretization. Then Theorem 3.1 shows that the fully implicit discretization is monotone. So we need to prove the discretization is stable. Set

$$U_{\max}^n = \max(\max_i U_i^n), g^n, h^n), \quad U_{\min}^n = \min(\min_i U_i^n), g^n, h^n).$$

where g^n, h^n is the boundary value of the nth times step. Using the same mathod as in Lemma 3.4, we have the more exact results:

$$U_{\min}^n \leq u_i^{n+1} \leq U_{\max}^n$$

Thus, we complete the proof.

3.4 The Superlinear Expectation

For reader convenience, we still use the same notions as in sublinear expectation (G-expectation), and give the main results of superlinear expectation.

Definition 3.5 *A superlinear expectation $\hat{\mathbb{E}}$ on $\mathcal{H}$ is a functional $\hat{\mathbb{E}} : \mathcal{H} \to R$ satisfying the following properties: for all $X, Y \in \mathcal{H}$, we have*

(a) Monotonicity: If $X \geq Y$ then $\hat{\mathbb{E}}[X] \geq \hat{\mathbb{E}}[Y]$;

(b) Constant preservation: $\hat{\mathbb{E}}[c] = c$;

(c) Sub-additivity: $\hat{\mathbb{E}}[X+Y] \leq \hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[Y]$;

(d) Positive homogeneity: $\hat{\mathbb{E}}[\lambda X] = \lambda \hat{\mathbb{E}}[X]$ for each $\lambda \geq 0$.

$(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ *is called a sublinear expectation space.*

The bounded boundary problem is

$$
\begin{aligned}
&\partial_t u - \tfrac{1}{2}(\underline{\sigma}^2 (D^2_{xx}u)^+ - \bar{\sigma}^2 (D^2_{xx}u)^-) = 0, \\
&u(0,x) = \varphi(x), \quad x \in [a,b], \\
&u(t,a) = g(t),\ u(t,b) = h(t) \ \ t \in [0,T].
\end{aligned}
\tag{3.15}
$$

where φ, g, h are continuous functions.

The equation (3.15) can be discretized by a standard finite difference method with variable timeweighting to give

$$
u_i^{n+1} - u_i^n = \quad \theta \alpha_i^n [u_{i+1}^n - 2u_i^n + u_{i-1}^n] + (1-\theta)\alpha_i^{n+1}[u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}], \tag{3.16}
$$

where

$$
\begin{aligned}
\alpha_i^n &:= \frac{\sigma(\Gamma_i^n)^2 \triangle t_i}{2(x_{i+1}-x_i)(x_i - x_{i-1})}, \\
\sigma(\Gamma_i^n) &:= \begin{cases} \underline{\sigma}, & \text{if } \Gamma_i^n \geq 0 \\ \bar{\sigma}, & \text{if } \Gamma_i^n < 0 \end{cases}, \\
\Gamma_i^n &:= \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(x_{i+1}-x_i)(x_i - x_{i-1})}.
\end{aligned}
\tag{3.17}
$$

In this paper, we consider the fully implicit schemes with $\theta = 0$, i.e.,

$$
u_i^{n+1} - u_i^n = \quad \alpha_i^{n+1}[u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}], \tag{3.18}
$$

The set of algebraic equation (3.15) is nonlinear for the formula of $\sigma(\Gamma_i^n)$. So we consider the discrete equation at each node as

$$
\varphi_i^n := u_i^n - u_i^{n+1} + \alpha_{i+1}^{n+1}[u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}]
$$

For further analysis the Newton iteration, we rewrite the discrete equation (3.16) in matrix form. Let

$$
\begin{aligned}
&U^{n+1} = [u_0^{n+1}, u_1^{n+1}, \cdots, u_m^{n+1}]', \quad U^n = [u_0^n, u_1^n, \cdots, u_m^n]', \\
&[M^n U^n]_i := -\alpha_i^n [u_{i+1}^n - 2u_i^n + u_{i-1}^n].
\end{aligned}
$$

$$
[I + M^{n+1}]U^{n+1} = U^n, \tag{3.19}
$$

Theorem 3.6 *The fully implicit discretization (3.19) convergences to the solution of the equation (3.15), as* $\triangle t,\ \triangle x \to 0$.

4 Numerical Example

In this section, we give an example which is important for financial market. The nonlinear probability $u(t,x) := \hat{\mathbb{E}}[I_{x+\sqrt{t}X\leq y}]$, $(t,x)\in[0,\infty)\times R$, should be the viscosity solution of the following G-heat equation:

$$\partial_t u - G(D^2_{xx}u) = 0,\ \ u(0,x) = I_{x\leq y}. \tag{4.1}$$

i.e.,

$$\begin{aligned} &\partial_t u - \tfrac{1}{2}(\bar{\sigma}^2(D^2_{xx}u)^+ - \underline{\sigma}^2(D^2_{xx}u)^-) = 0, \quad x\in R, \quad t>0, \\ &u(0,x) = I_{x\leq y}, \quad x\in R, \end{aligned} \tag{4.2}$$

Next, we consider a boundary problem of (4.2), i.e.,

$$\begin{aligned} &\partial_t u - \tfrac{1}{2}((D^2_{xx}u)^+ - \tfrac{1}{4}(D^2_{xx}u)^-) = 0, \\ &u(0,x) = I_{x\leq 0}, \quad x\in[-9.99,10], \\ &u(t,-9.99) = 1,\ \ u(t,10) = 0, \quad t\in[0,1]. \end{aligned} \tag{4.3}$$

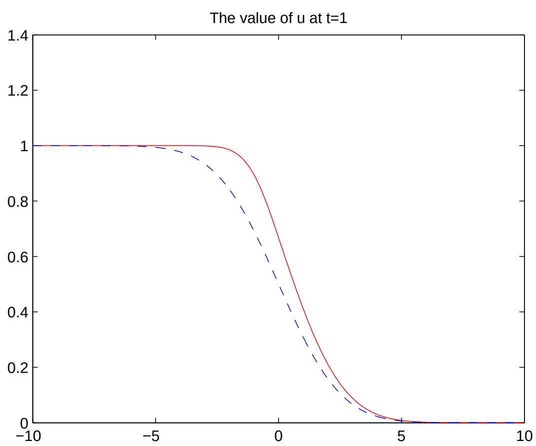
For a given probability space $(\Omega,\mathcal{F},P)$, the linear probability $u(t,x) := E[I_{x+\sqrt{t}X\leq y}]$, $(t,x)\in[0,\infty)\times R$, should be the viscosity solution of the following heat equation:

$$\partial_t u - D^2_{xx}u = 0,\ \ u(0,x) = I_{x\leq y}. \tag{4.4}$$

We also consider a boundary problem of (4.4), i.e.,

$$\begin{aligned} &\partial_t \hat{u} - \tfrac{1}{2}((D^2_{xx}\hat{u})^+ - (D^2_{xx}\hat{u})^-) = 0, \\ &\hat{u}(0,x) = I_{x\leq 0}, \quad x\in[-9.99,10], \\ &\hat{u}(t,-9.99) = 1,\ \ \hat{u}(t,10) = 0, \quad t\in[0,1]. \end{aligned} \tag{4.5}$$

Comparing the value of $u(1,x)$ and $\hat{u}(1,x), x\in[-9.99,10]:$



The red line is the value of function $u(1,x)$, $x \in [-9.99, 10]$, and the blue line is the the value of function $\hat{u}(1,x)$, $x \in [-9.99, 10]$. By $u(t,x) := \hat{\mathbb{E}}[I_{x+\sqrt{t}X\leq y}]$ and $\hat{u}(1,x) = E[I_{x+\sqrt{t}X\leq y}]$, we have

$$\hat{\mathbb{E}}[I_{X\leq 0}] = u(1,0) = 0.6680, \;\; P(X \leq 0) = u(1,0) = 0.5010.$$

These results show that the G-expectation have more improtant application in financial market (see [2])

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